Stat 155 Lecture 15 Notes

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1 Correlated Equilibria and Braess's Paradox

1.1 An example of inefficient Nash equilibria

Example 1.1. Consider an example of traffic, where two drivers have to decide whether to stop or go. Stopping has a cost of 1, and going has a payoff of 1. However, if both cars go, they crash, and the cost is 100 to each driver. The payoff bimatrix is

	Go	Stop
Go	$(-100, -100)$	$(1, -1)$
Stop	$(-1, 1)$	$(-1, -1)$

The pure Nash equilibria are (go, stop) and (stop, go). To find mixed Nash equilibriam we solve

$$\begin{pmatrix} -100 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} y \\ 1-y \end{pmatrix} = \begin{pmatrix} v_1 \\ v_1 \end{pmatrix},$$

which gives the Nash equilibrium $((2/101, 99/101), (2/101, 99/101))$. Under the mixed Nash equilibrium, each player gets a payoff of -1. Can we do better?

Here is a better solution. Suppose there is a traffic signal with

$$\mathbb{P}((\text{Red}, \text{Green})) = \mathbb{P}((\text{Green}, \text{Red})) = 1/2,$$

and both players agree that Red means Stop and Green means Go. After they both see the traffic signal, the players have no incentive to deviate from the agreed actions. The expected payoff for each player is 0, higher than that of the mixed Nash equilibrium.

1.2 Correlated strategy pairs and equilibria

Definition 1.1. For a two player game with strategy sets $S_1 = \{1, \dots, m\}$ and $S_2 = \{1, \dots, n\}$, a *correlated strategy pair* is a pair of random variables (R, C) with some joint probability distribution over pairs of actions $(i, j) \in S_1 \times S_2$.

Example 1.2. In the traffic example, the traffic light induces a correlated strategy pair with joint distribution

	Go	Stop
Go	0	1/2
Stop	1/2	0

Compare this definition with a pair of mixed strategies. Let $x \in \Delta_{S_m}$ and $y \in \Delta_{S_n}$ such that $\mathbb{P}(R = i) = x_i$ and $\mathbb{P}(C = j) = x_j$. Then, choosing the two actions (R, C) independently gives $\mathbb{P}(R = i, C = j) = x_i y_j$.

In the traffic signal example, we cannot have $\mathbb{P}(\text{Stop}, \text{Go}) > 0$ and $\mathbb{P}((\text{Go}, \text{Stop}) > 0$ without $\mathbb{P}(\text{Go}, \text{Go}) > 0$.

Definition 1.2. A correlated strategy pair for a two-player game with payoff matrices A and B is a *correlated equilibrium* if

1. $\forall i, i' \in S_1$, $\mathbb{P}(R = i) > 0 \implies \mathbb{E}[a_{i,C} \mid R = i] \geq \mathbb{E}[a_{i',C} \mid R = i]$.
2. $\forall j, j' \in S_2$, $\mathbb{P}(C = j) > 0 \implies \mathbb{E}[b_{R,j} \mid C = j] \geq \mathbb{E}[b_{R,j'} \mid C = j]$.

Compare this with Nash equilibria. Let $(x, y) \in \Delta_{S_m} \times \Delta_{S_n}$ be a strategy profile, and let R and C be independent random variables with $X_i = \mathbb{P}(R = i)$ and $\mathbb{P}(C = j) = y_j$. Then (x, y) is aNash equilibrium iff

1. $\forall i, i' \in S_1$, $\mathbb{P}(R = i) > 0 \implies \mathbb{E}[a_{i,C}] \geq \mathbb{E}[a_{i',C}]$.
2. $\forall j, j' \in S_2$, $\mathbb{P}(C = j) > 0 \implies \mathbb{E}[b_{R_j}] \geq \mathbb{E}[b_{R,j'}]$.

This is because

$$\mathbb{E}[a_{i,C}] = \sum_{j \in S_2} a_{i,j} \mathbb{P}(C = j) = \sum_{j \in S_2} a_{i,j} y_j = e_i^\top A y,$$

coupled with the principle of indifference. Since R and C are independent, these expectations and the conditional expectations are identical. Thus, a Nash equilibrium is a correlated equilibrium.

Example 1.3. Consider the pair of random variables (R, C) with joint distribution

$$\begin{pmatrix} 0 & 1/3 \\ 1/3 & 1/3 \end{pmatrix},$$

so $\mathbb{P}(\text{Go}, \text{Go}) = 0$, and $\mathbb{P}(\text{Go}, \text{Stop}) = \mathbb{P}(\text{Stop}, \text{Go}) = \mathbb{P}(\text{Stop}, \text{Stop}) = 1/3$.

Is this a correlated equilibrium for the traffic example? We need to check if

$$\mathbb{E}[a_{\text{Stop},C} \mid R = \text{Stop}] \geq \mathbb{E}[a_{\text{Go},C} \mid R = \text{Stop}],$$

$$\mathbb{E}[a_{\text{Gp},C} \mid R = \text{Go}] \geq \mathbb{E}[a_{\text{Stop},C} \mid R = \text{Go}],$$

$$\mathbb{E}[b_{R,\text{Stop}} \mid C = \text{Stop}] \geq \mathbb{E}[b_{R,\text{Go}} \mid C = \text{Stop}],$$
$$\mathbb{E}[b_{R,\text{Gp}} \mid C = \text{Go}] \geq \mathbb{E}[b_{R,\text{Stop}} \mid C = \text{Go}].$$

Notice that $\mathbb{P}(C = \text{Go} \mid R = \text{Stop}) = 1/2$, so

$$\mathbb{E}[a_{\text{Stop},C} \mid R = \text{Stop}] = -1 > -99/2 = \mathbb{E}[a_{\text{Go},C} \mid R = \text{Stop}].$$

Also, $\mathbb{P}(C = \text{Go} \mid R = \text{Go}) = 0$, so

$$\mathbb{E}[a_{\text{Go},C} \mid R = \text{Go}] = 1 > -1 = \mathbb{E}[a_{\text{Stop},C} \mid R = \text{Go}].$$

What is the expected payoff for each player? For Player 1, it is

$$\mathbb{E}[a_{R,C}] = \frac{1}{3} a_{\text{Go},\text{Stop}} + \frac{1}{3} a_{\text{Stop},\text{Go}} + \frac{1}{3} a_{\text{Stop},\text{Stop}} = -\frac{1}{3}.$$

For Player 2, it is the same.

1.3 Interpretations and comparisons to Nash equilibria

How do correlated equilibria compare to Nash equilibria? Nash's Theorem implies that there is always a correlated equilibrium. They are also easy to find via linear programming. It is not unusual for correlated equilibria to achieve better solutions for both players than Nash equilibria, as in the traffic example.

We can think of a correlated equilibrium being implemented in two equivalent ways:

1. There is a random draw of a correlated strategy pair with a known distribution, and the players see their strategy only.

2. There is a draw of a random variable (an 'external event') with a known probability distribution, and a private signal is communicated to the players about the value of the random variable. Each player chooses a mixed strategy that depends on this private signal (and the dependence is common knowledge).

Given any two correlated equilibria, you can combine them to obtain another: Imagine a public random variable that determines which of the correlated equilibria will be played. Knowing which correlated equilibrium is being played, the players have no incentive to deviate. The payoffs are convex combinations of the payoffs of the two correlated equilibria.

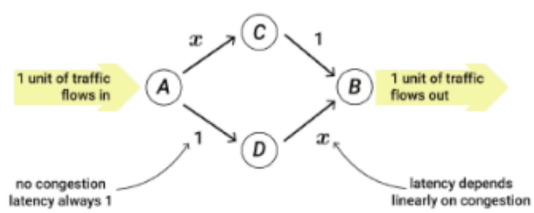
1.4 Braess's paradox

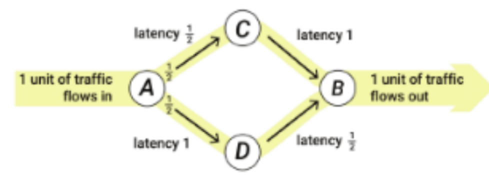
In 2009, New York City closed Broadway at Times Square with the aim of reducing traffic congestion. It was successful. It seems counterintuitive that removing options for transportation would reduce traffic congestion. But there are other examples, as well:

- In 2005, the Cheonggyecheon highway was removed, speeding up traffic in downtown Seoul, South Korea.
- 42nd Street in NYC closed for Earth Day. Traffic improved.
- In 1969, congestion decreased in Stuttgart, West Germany, after closing a major road.

Why does this happen? Drivers, acting rationally, seek the fastest route, which can lead to bigger delays (on average, and even for everyone).

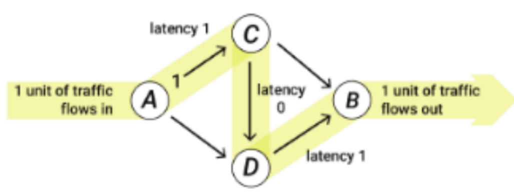
Example 1.4. Consider the following network from destination A to B, where the latency of traffic on each edge is dependent on the proportion of the traffic flow traveling along that edge.



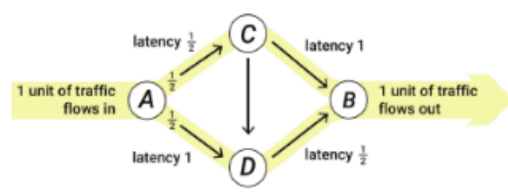


The optimal flow is for 1/2 of the traffic to travel though C and 1/2 of the traffic to travel though D.

What happens when we add an edge from C to D?



Nash equilibrium flow



optimal flow

A Nash equilibrium flow has all of the traffic travel to C, then to D, and then to B. This has a latency of 2 for every driver, as opposed to the optimal form from before, which only had a latency of 3/2 for each driver. So adding edges is not always efficient.